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PRESSURE RECOVERY CURVE IN THE RELAXATION THEORY OF FILTRATION FOR A CONTINUOUS SPECTRUM OF RELAXATION TIMES

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Microscopic relaxation processes in the system "a rock–a saturated fluid" manifest themselves in unsteady filtration processes, in which the characteristic time of the macroscopic process is comparable with internal times of relaxation. They should be taken into account in interpreting the corresponding tests.

Asymptotic relations for the pressure recovery curve (PRC) have been obtained previously for the relaxation theory of filtration at small times [1] and for large times of the discrete spectrum of internal relaxation times [2]. In the present paper, we derive asymptotic relations for the PRC at large times for the continuous spectrum of purely dissipative internal relaxation processes.

For an arbitrary function of time $f = f(t)$, we denote its Fourier transform by $f_F = f_F(\omega)$:

$$f_F(\omega) = \int_{-\infty}^{+\infty} e^{-i\omega t} f(t)\, dt.$$

We give briefly the main statements of the theory of relaxation isothermal filtration in a homogeneous isotropic reservoir [2–7].

In the relaxation theory of filtration, the generalized Darcy law is adopted:

$$u^i(t_0, x^j) = -k\mu^{-1} \int_{-\infty}^{+\infty} K(t_0 - t)\, \frac{\partial G}{\partial x^i}(t, x^j)\, dt. \qquad (1)$$

Here $G = p + \rho U$, u^i is the filtration velocity, k is the permeability, μ is the shear viscosity of the fluid, p is the pressure, ρ is the mass density, U is the gravitational potential; the superscripts i and j take values 1, 2, and 3, which correspond to Cartesian coordinates x^i.

The kernel $K = K(t)$ characterizes internal relaxation processes in the fluid-saturated porous medium. It obeys a number of conditions that follow from physical and thermodynamic considerations:

1. $K(t)$ is a nonnegative, monotonically decreasing function of dimension t^{-1} (t is time).
2. $\int_{-\infty}^{+\infty} K(t)\, dt = 1$ is the condition for reducing (1) to the Darcy law for slow processes.
3. $K(t) = 0$ for $t < 0$ (causality); $K(0) < +\infty$ is the condition for a finite signal velocity [8].

According to the Paley–Wiener theorem [9], condition 3 makes the function $K_F = K_F(\omega)$ holomorphic in the lower half of the complex plane. It has been shown [6, 7] that the following dissipation condition holds.

4. Re $K_F(\omega) > 0$ for Im $\leqslant 0$.

From condition 2 it follows that

$$K_F(0) = 1. \qquad (2)$$

From condition 3 it follows that the asymptotic relation

$$K_F(\omega) = k_1 (i\omega)^{-1} + o(|\omega|^{-1}), \quad k_1 = K(0) \qquad (3)$$

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is valid in the holomorphic region.

In the present paper, we consider the case where the fluid-saturated porous medium has a continuous spectrum of purely dissipative internal relaxation processes. This means that the kernel can be represented in the functional form

$$K(t) = \int_0^{+\infty} A(\tau)\,\tau^{-1} \exp(-t/\tau)\,d\tau, \tag{4}$$

where $A(\tau)$ is a smooth non-negative function. In [2], formally the same functional form (4) was used for the kernel, but the weight function $A(\tau)$ was written as the sum of δ-functions. The Fourier transform for relation (4) takes the form

$$K_F(\omega) = \int_0^{+\infty} A(\tau)(1 + i\tau\omega)^{-1} d\tau. \tag{5}$$

From (2) and (5) we obtain the normalized equality

$$1 = \int_0^{+\infty} A(\tau)\,d\tau. \tag{6}$$

Furthermore, from condition 3 follows the convergence of the integral

$$k_1 = \int_0^{+\infty} \tau^{-1} A(\tau)\,d\tau < +\infty. \tag{7}$$

It is easy to verify that, if relations (4)–(7) are adopted, conditions 1–4 for the relaxation kernel are satisfied. From relation (5) it follows that the function $K_F(\omega)$ is holomorphic in the complex plane with a cut along the ray $\mathrm{Re}\,\omega = 0$, $\mathrm{Im}\,\omega > 0$. Using the Sokhotskii–Plemel formula, we can calculate the function $K_F(\omega)$ at the cut borders:

$$\begin{gathered} K_{F+} = K_F(iy + \varepsilon) = L_1(y) - i\pi L_2(y), \quad K_{F-} = K_F(iy - \varepsilon) = L_1(y) + i\pi L_2(y), \\ L_1(y) = \mathrm{V.\,p.} \int_0^{+\infty} z^{-1} A(z^{-1})(z - y)^{-1} dz, \quad L_2(y) = y^{-1} A(y^{-1}). \end{gathered} \tag{8}$$

Here and below, $y > 0$ and ε is a small positive number.

As in [2], we consider the linear problem of the PRC in a cylindrical symmetrical formulation. The dynamics of the pressure field is determined by the integrodifferential equation [2]

$$\frac{\partial}{\partial t} p(t_0, r) = æ \int_{-\infty}^{+\infty} K(t_0 - t)\,\Delta p(t, r)\,dt, \tag{9}$$

where $æ = kE/(m\mu)$, r is the distance from the well axis, $\Delta = \partial^2/\partial r^2 + r^{-1}\partial/\partial r$ is the Laplacian, m is the porosity, $E = (E_1^{-1} + (m^{-1} - 1)E_2^{-1})^{-1}$, and E_1 and E_2 are the elastic bulk moduli for the fluid and the solid phase (the skeleton), respectively. The parameter r varies within the range $r_1 \leqslant r \leqslant r_2$, where r_1 is the well radius, and r_2 is the recharge radius.

For Eq. (9), it is necessary to adopt two boundary conditions [2]:

$$q(t) = \lambda \int_{-\infty}^{+\infty} K(t_0 - t) \frac{\partial}{\partial r} p(t, r_1)\,dt, \quad \lambda = 2\pi r_1 \rho_0 \mu^{-1}; \tag{10}$$

$$p(t, r_2) = p_b. \tag{11}$$

Here ρ_0 is the mass density of the fluid bed, $q(t)$ is the mass yield per unit productive thickness of the bed, and p_b is the bed pressure.

We shall use a system of units in which

$$\text{æ} = r_1 = 1. \tag{12}$$

The quantity æ has the dimension of l^2/t, where l is the length, and, hence, condition (12) fixes units of length and time.

We now introduce a new unknown function $P = P(t, r) = p(t, r) - p_b$.

Taking the Fourier transform of (9)–(11), we obtain the ordinary second-order differential equation

$$(\Delta - \alpha^2)\, P_F = 0 \tag{13}$$

with the boundary conditions

$$q_F = \lambda K_F \left.\frac{\partial P_F}{\partial r}\right|_{r=1}, \qquad \left.P_F\right|_{r=r_2} = 0. \tag{14}$$

Equation (13) now includes a new complex function $\alpha = \alpha(\omega)$, which is determined from the relations

$$\alpha^2 = i\omega/K_F(\omega), \qquad \operatorname{Re}\alpha \geqslant 0. \tag{15}$$

Using the general conditions 1–4 and without invoking the explicit formula (5), we have previously shown [2] that the function $\alpha = \alpha(\omega)$ is holomorphic for $\operatorname{Im}\omega < 0$ and continuous up to the real axis. Formula (5) allows one to continue the function $\alpha = \alpha(\omega)$ to the upper half of the complex plane. Obviously, here it has singularities related to the zeros and singularities of the function $K_F(\omega)$, and also with the procedure of extracting the root in (15). Note that

$$\operatorname{Im} K_F = -A_1 \operatorname{Re}\omega, \quad A_1 = \int_0^{+\infty} \tau A(\tau)|1 + i\tau\omega|^{-2}\, d\tau > 0.$$

Therefore, the function $K_F(\omega)$ does not vanish for $\operatorname{Re}\omega \neq 0$, and, hence, the function $\alpha(\omega)$ is holomorphic with a cut along the ray $\operatorname{Re}\omega = 0$, $\operatorname{Im}\omega > 0$. It is not hard to calculate its values at the cut borders:

$$\alpha_+ = \alpha(iy + \varepsilon) = iy^{1/2}(K_{F+})^{-1/2}; \tag{16}$$

$$\alpha_- = \alpha(iy - \varepsilon) = -iy^{1/2}(K_{F-})^{-1/2}. \tag{17}$$

Problem (13), (14) has the solution

$$P_F = \frac{q_F(-I_0(\alpha r_2)K_0(\alpha r) + K_0(\alpha r_2)I_0(\alpha r))}{\lambda K_F \alpha (K_0(\alpha r_2)I_1(\alpha) + K_1(\alpha)I_0(\alpha r_2))}, \tag{18}$$

where $K_n(z)$ and $I_n(z)$ are MacDonald functions [10].

If the well operates with a constant yield Q, we have $P = P(r) = \lambda^{-1}Q\ln(r/r_2)$.

To solve the PRC problem, we must set $q(t) = Q\theta(-t)$ in (18), where $\theta(t)$ is the Heaviside function. We introduce the difference between the current and initial pressure: $\Phi = P - \lambda^{-1}Q\ln(r/r_2)$.

The function Φ_F can be calculated from formula (18) if we substitute $q_F = iQ(\omega - i\varepsilon)^{-1}$, which, in real time, formally corresponds to the yield $q(t) = -Q\theta(t)$. As in [2], we shall seek an intermediate asymptotic expression of the PRC for which the finiteness of r_2 can be ignored. In the limit $r_2 \to +\infty$, relation (18) reduces to the form $\Phi_F = -q_F K_0(\alpha r)/(\lambda K_F \alpha K_1(\alpha))$.

To determine the PRC, it is necessary to calculate the function $\varphi(t) = \left.\Phi\right|_{r=1}$. Taking the inverse Fourier transform of this function, we obtain the relation

$$\varphi(t) = -(2\pi\lambda)^{-1}Qi \int_{-\infty}^{+\infty} (\omega - i\varepsilon)^{-1} f_1(\omega) e^{i\omega t}\, d\omega \qquad \left[f_1(\omega) = \frac{K_0(\alpha)}{K_F \alpha K_1(\alpha)}\right]. \tag{19}$$

Formula (19) is the PRC in the form of a functional of the kernel K:

$$\varphi(t) = \Psi[t; K(t')]. \tag{20}$$

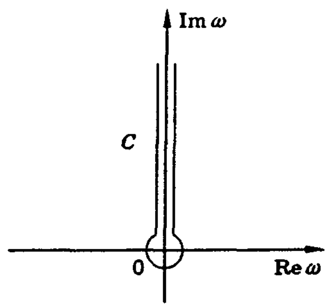

Fig. 1

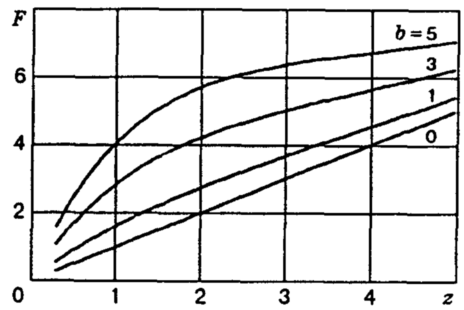

Fig. 2

As in [2], we shall seek the asymptotic $\varphi(t)$ for large t. If t far exceeds the internal relaxation times, the effect of the relaxation kernel becomes insignificant. Therefore, if we simply pass to the limit $t \to +\infty$ in formula (19), we obtain the usual logarithmic asymptotic relation [11] that corresponds to the case of a trivial kernel $K_F = 1$. To find the asymptotic relation for large t comparable in order of magnitude with the internal relaxation times, in functional (19) we should perform the substitution

$$t = t_*\delta^{-1}, \qquad K(t') = K_*(t'_*) \tag{21}$$

and, considering t_* and K_* fixed, compute the asymptotic expression for $\delta \to +0$. As $\delta \to +0$, functional (20) increases infinitely. In the space of functionals of the form of (20), we introduce the equivalence ratio: the functionals Ψ_1 and Ψ_2 are equivalent ($\Psi_1 \sim \Psi_2$) if the substitution of (21) in the expressions for the functionals for $\delta \to +0$ leads to

$$\Psi_1 - \Psi_2 = O(1).$$

Going over to the equivalent functionals, we simplify formula (19). For this, recall that the MacDonald functions admit the representation [10]

$$K_0(z) = -J_0(z^2)\ln(z/2) + W_0(z^2), \quad K_1(z) = z^{-1}(J_1(z^2)\ln(z) + W_1(z^2)), \tag{22}$$

where $J_n(z)$ and $W_n(z)$ are integer functions, $J_0(0) = W_1(0) = 1$, $J_1(0) = 0$, $W_0(0) = -\mathrm{C}$, and C is the Euler constant. Formulas (22) allows one to distinguish the asymptotic expression of the function $f_1(\omega)$ by substitution (21). In this case, we obtain

$$\varphi(t) \sim (2\pi\lambda)^{-1}Qi \int_{-\infty}^{+\infty} (\omega - i\varepsilon)^{-1} f_2(\omega) e^{i\omega t} d\omega, \qquad f_2(\omega) = \ln(\alpha/K_F). \tag{23}$$

We transform the integral along the real axis in formula (23) to the integral along contour C (Fig. 1). Uncovering the integrands with allowance for (8), (16), and (17), we obtain

$$\varphi(t) \sim (2\pi\lambda)^{-1}Qi(i\pi\ln\varepsilon + I_{1\varepsilon} + I_{2\varepsilon}), \qquad I_{1\varepsilon} = i\pi\int_{\varepsilon}^{+\infty} y^{-1}e^{-yt}\,dy,$$
$$I_{2\varepsilon} = \int_{\varepsilon}^{+\infty} y^{-1}((\ln\alpha_+/K_{F+}) - (\ln\alpha_-/K_{F-}) - i\pi)e^{-yt}dy. \tag{24}$$

Here ε is the radius of an infinitely small circumference along which the point $\omega = 0$ is handled (Fig. 1). For the passage to the limit $\varepsilon \to 0$ in formula (24), it is reasonable to use two auxiliary formulas from [12]: formula No. 3.352.4

$$\int_0^{+\infty} \frac{\exp(-bz)\,dz}{a+z} = -\exp(ab)\mathrm{Ei}(-ab) \qquad (a, b > 0) \tag{25}$$

and formula No. 8.214.1

$$\mathrm{Ei}(z) = \mathrm{C} + \ln(-z) + \sum_{n=1}^{\infty} z^n (nn!)^{-1} \quad (z < 0). \tag{26}$$

Note that the integral $I_{2\varepsilon}$ converges as $\varepsilon \to 0$. The limit $I_{1\varepsilon}$ as $\varepsilon \to 0$ is calculated based on formulas (25) and (26). As a result, relation (24) takes a form that does not contain the parameter ε:

$$\varphi(t) \sim (2\pi\lambda)^{-1} Q(\pi \ln t + \pi \ln \mathrm{C} + i I_{20}). \tag{27}$$

We can apply substitution (21) to the functional I_{20} and calculate the main term of the asymptotic expression as $\delta \to 0$:

$$\begin{gathered} I_{20} \sim (-i \ln t J(t)), \\ J(t) = (2i)^{-1} \int_0^{+\infty} y^{-1} (K_{F+}^{-1} - K_{F-}^{-1})\, \mathrm{e}^{-yt}\, dy. \end{gathered} \tag{28}$$

From (27) we finally obtain the desired asymptotic expression for the PRC:

$$\varphi(t) \sim (2\lambda)^{-1} Q(1 + \pi^{-1} J(t)) \ln t. \tag{29}$$

Using formulas (8), it is not hard to verify that $J(t)$ is a positive function:

$$J(t) = \pi \int_0^{+\infty} y^{-2} A(y^{-1}) |K_{F+}|^{-2} \mathrm{e}^{-yt} dy > 0. \tag{30}$$

On the other hand, treating relation (28) as an integral in the complex plane along the contour C (Fig. 1), we can rearrange it with allowance for (3):

$$\pi^{-1} J(t) = (2\pi)^{-1} \int_{-\infty}^{+\infty} ((i\omega - \varepsilon)^{-1} K_F^{-1} - k_1^{-1}) \mathrm{e}^{i\omega t} d\omega.$$

The right side of this equation is the inverse Fourier transform of the function $((i\omega K_F)^{-1} - k_1^{-1})$.

Thus, internal relaxation processes lead to the appearance of the function $J(t)$ in the expression for the PRC (29). This function vanishes for the trivial kernel $K_F = 1$. By virtue of inequality (30), neglect of relaxation phenomena in interpreting experimental PRC leads to underestimated permeability k.

To constructively use formula (29) in interpreting PRC, one should specify the function $J(t)$. Note that the asymptotic behavior of $J(t)$ for large t is determined by the asymptotic behavior of the weight function $A(\tau)$ for large relaxation times. We assume that, for large τ, we have the exponential spectrum

$$A(\tau) \approx a_0 \tau^{-1-\beta}, \qquad 0 < \beta < 1. \tag{31}$$

Assumption (31) satisfies the condition of convergence of integral (6). From (30) and (31) we find the asymptotic expression of $J(t)$ for large t:

$$J(t) \approx \pi a_1 t^{-\beta}, \qquad a_1 = a_0 \Gamma(\beta). \tag{32}$$

Substituting the asymptotic expression (32) into (29), we obtain a formula for the PRC for the exponential spectrum of internal relaxation times. In comparison with the classical formula for the PRC [11]

$$\Delta p = (2\lambda)^{-1} Q \ln(t/t_0), \tag{33}$$

where t_0 is a constant with the dimension of time, the new formula contains an additional factor of the form $(1 + a_1 t^{-\beta})$ and two additional "adjusting" parameters: β and a_1.

As a graphical illustration of the result obtained, Fig. 2 shows a family of the curves of $F = (1 + b(t/t_0)^{-\beta}) \ln(t/t_0)$ versus $z = \ln(t/t_0)$ for $\beta = 1/2$ at various values of the parameter b. The case $b = 0$ qualitatively corresponds to the classical PRC (33).

As can be seen from the results obtained, for an arbitrary spectral function $A(\tau)$, the problem of determining the filtration-capacity properties of a bed from the PRC becomes incorrect. For a particular type of spectral function with a finite number of free parameters, the problem of interpreting the PRC can be correct, but this problem requires additional investigation.

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